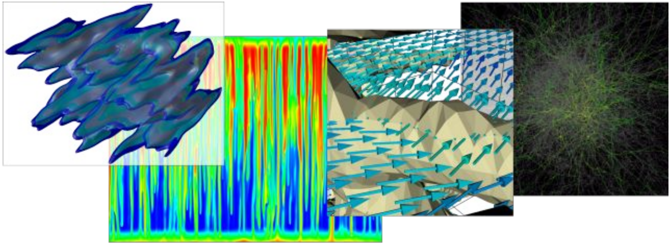
DUNE PDELab Tutorial 02

The Cell-centered Finite Volume Method



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Motivation

This tutorial extends on tutorial 00 by

1) Solving a **nonlinear** stationary PDE
2) Using **different types of boundary conditions**
3) Implementing a **cell-centered finite volume method with two-point flux approximation** as an example of a non-conforming scheme.
4) Implementing **all possible methods of a local operator**.

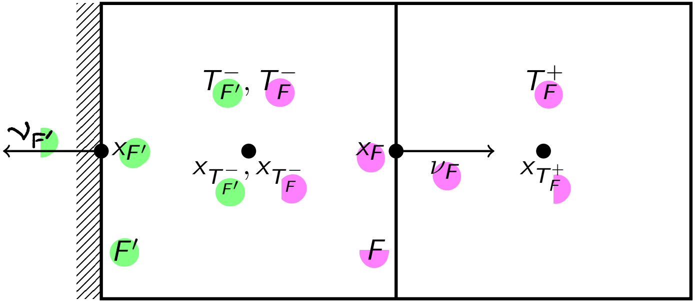
PDE Problem

We consider the problem (same as in tutorial 1)

$$
\begin{aligned}
-\Delta u + q(u) &= f && \text{in } \Omega,\\
u &= g && \text{on } \Gamma_D \subseteq \partial\Omega,\\
-\nabla u \cdot \nu &= j && \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_D.
\end{aligned}
$$

- $q : \mathbb{R} \to \mathbb{R}$ a nonlinear function
- $f : \Omega \to \mathbb{R}$ the source term
- $g : \Omega \to \mathbb{R}$ a function for Dirichlet boundary conditions on Γ_D
- $j : \Gamma_N \to \mathbb{R}$ a function for Neumann (flux) boundary conditions
- ν: unit outer normal to the domain

Notation for Interior Intersections



$\mathcal{F}_h^i = \{F_1, \ldots, F_N\}$: set of *interior intersections*

$\mathcal{F}_h^{\partial\Omega} = \{F_1, \ldots, F_L\}$: set of *boundary intersections* independent of boundary condition

$\mathcal{F}_h^{\partial\Omega} = \mathcal{F}_h^{\Gamma_D} \cup \mathcal{F}_h^{\Gamma_N}$

Discrete Weak Formulation

Finite volume methods use the function space

$$W_h = \{w \in L^2(\Omega) \;:\; w|_T = \text{const for all } T \in \mathcal{T}_h\}.$$

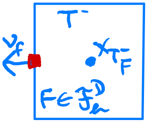
then we can derive the following discrete weak formulation: For any $v \in W_h$:

$$\int_\Omega fv\,dx = \int_\Omega [-\Delta u + q(u)]v\,dx = \sum_{T\in\mathcal{T}_h} v \int_T -\Delta u + q(u)\,dx \qquad (v \text{ const on } T)$$

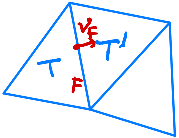
$$= \sum_{T\in\mathcal{T}_h} \left[\int_T q(u)v\,dx - \int_{\partial T} \nabla u \cdot \nu v\,ds\right] \qquad \text{(Gauss' thm.)}$$

$$= \sum_{T\in\mathcal{T}_h} \int_T q(u)v\,dx - \sum_{F\in\mathcal{F}_h^i} \int_F \underbrace{\nabla u \cdot \nu_F}_{=\frac{\partial u}{\partial \nu_F}} \overbrace{\left[v(x_{T_F^-}) - v(x_{T_F^+})\right]}^{\text{„jump" term}} ds$$

$$- \sum_{F\in\mathcal{F}_h^{\partial\Omega}} \int_F \nabla u \cdot \nu_F\,ds. \qquad \text{(rearrange)}$$



$\mathcal{F}_h^{\partial\Omega} = \mathcal{F}_h^D \cup \mathcal{F}_h^N$

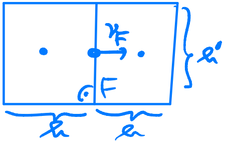


Finite Volume Scheme

$\rightsquigarrow$ discrete case : seek $u_h \in W_h$

Now approximate the directional derivative

$$\nabla u \cdot \nu_F \approx \frac{u_h(x_{T_F^+}) - u_h(x_{T_F^-})}{\|x_{T_F^+} - x_{T_F^-}\|}, \qquad \text{(two-point flux approximation)}$$



and all integrals by the midpoint rule

$$\int_T f\,dx \approx f(x_T)|T|$$

to get the abstract problem

$$\boxed{\text{Find } u_h \in W_h \text{ s.t.:} \quad r_h^{\mathrm{CCFV}}(u_h, v) = 0 \quad \forall v \in W_h}$$

where the residual form is ...

Residual Form

$$r_h^{\mathrm{CCFV}}(u_h, v) = \sum_{T\in\mathcal{T}_h} q(u_h(x_T))v(x_T)|T| - \sum_{T\in\mathcal{T}_h} f(x_T)v(x_T)|T| \qquad \text{1. and 2.}$$

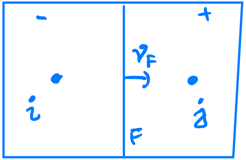
$$- \sum_{F\in\mathcal{F}_h^i} \frac{u_h(x_{T_F^+}) - \overset{z_i \varphi_i}{u_h}(x_{T_F^-})}{\|x_{T_F^+} - x_{T_F^-}\|}\left[v(x_{T_F^-}) - v(x_{T_F^+})\right]|F| \qquad \text{3.}$$

$$+ \sum_{F\in\mathcal{F}_h^{\partial\Omega}\cap\Gamma_D} \frac{u_h(x_{T_F^-})}{\|x_F - x_{T_F^-}\|} v(x_{T_F^-})|F| \qquad \text{4.}$$

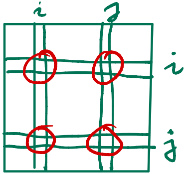
$$- \sum_{F\in\mathcal{F}_h^{\partial\Omega}\cap\Gamma_D} \frac{g(x_F)}{\|x_F - x_{T_F^-}\|} v(x_{T_F^-})|F| \qquad \text{5.}$$

$$+ \sum_{F\in\mathcal{F}_h^{\partial\Omega}\cap\Gamma_N} j(x_F)v(x_{T_F^-})|F|. \qquad \text{5.}$$

jacobian:



matrix A:



Remarks on the Residual Form

Five different types of integrals are involved in the residual form:

1. Volume integral depending on trial and test function.
2. Volume integral depending on test function only.
3. Interior intersection integral depending on trial and test function.
4. Boundary intersection integral depending on trial and test function.
5. Boundary intersection integral depending on test function only.

Dirichlet as well as Neumann boundary conditions are built weakly into the residual form!

No constraints on the function space are necessary in this case

Can be extended to discontinuous Galerkin methods

General Residual Form

A residual form in PDELab has the following structure:

$$
\begin{aligned}
r(u,v) = {} & \sum_{T\in\mathcal{T}_h} \alpha_T^V(R_T u, R_T v) + \sum_{T\in\mathcal{T}_h} \lambda_T^V(R_T v) \\
& + \sum_{F\in\mathcal{F}_h^i} \alpha_F^S(R_{T_F^-} u, R_{T_F^+} u, R_{T_F^-} v, R_{T_F^+} v) \\
& + \sum_{F\in\mathcal{F}_h^{\partial\Omega}} \alpha_F^B(R_{T_F^-} u, R_{T_F^-} v) + \sum_{F\in\mathcal{F}_h^{\partial\Omega}} \lambda_F^B(R_{T_F^-} v).
\end{aligned}
$$

S = „skeleton"

which results in the following methods on the local operator

	volume	skeleton	boundary
residual	alpha_volume lambda_volume	alpha_skeleton	alpha_boundary lambda_boundary
Jacobian	jacobian_volume	jacobian_skeleton	jacobian_boundary
Jac. app.	jacobian_apply_volume	jacobian_apply_skeleton	jacobian_apply_boundary

→ There are up to 11 methods on the local operator.
The CCFV scheme implements them all!

Implementation Overview

The tutorial consist of the following files:

1) The ini-file `tutorial02.ini` holds parameters which control the execution.
2) The main file `tutorial02.cc` includes the necessary C++, DUNE and PDELab header files; contains the `main` function; instantiates DUNE grid objects and calls the `driver` function.
3) File `driver.hh` instantiates the PDELab classes for solving a nonlinear stationary problem with the cell-centered finite volume method and solves the problem.
4) File `nonlinearpoissonfv.hh` contains the class `NonlinearPoissonFV` realizing a PDELab local operator
5) File `problem.hh` contains a parameter class which encapsulates the user-definable part of the PDE problem

Same as in tutorial 1 !